Webology (ISSN: 1735-188X)
Volume 18, Number 6, 2021

Boundary Layer Fluid flow Problem Of Falkner-Skan Model With Wall Stretching And Transfer Of Mass Effects: Evolutionary Optimized Quartic Spline Approach

Fazle Subhan[1], Saeed Islam[1], Muhammad Asif Zahoor Raja[2,*], Iftikhar Uddin[1], Naveed Ishtiaq Chaudhary[2], Muhammad Shoaib[3], Kottakkaran Sooppy Nisar[4,*], Malak S. Alqahtani[5], Mohamed Abbas[6,7], C A Saleel[8]

[1] Department of Mathematics, Abdul Wali Khan University Mardan (KP), 23200, Pakistan.

[2] Future Technology Research Center, National Yunlin University of Science and Technology, 123 University Road, Section 3, Douliou, Yunlin 64002, Taiwan, R.O.C.

[3] AI center, Yuan Ze University, Taiwan.

[4] Department of Mathematics, College of Science and Humanities in Alkharj, Prince Sattam bin Abdulaziz University, Saudi Arabia.

[5] Computer Engineering Department, College of Computer Science, King Khalid University, Abha 61421, Saudi Arabia.

[6] Electrical Engineering Department, College of Engineering, King Khalid University, Abha 61421, Saudi Arabia.

[7] Computers and communications Department, College of Engineering, Delta University for Science and Technology, Gamasa 35712, Egypt.

[8] Mechanical Engineering Department, College of Engineering, King Khalid University, Abha 61421, Saudi Arabia.

* Correspondence

Abstract: This study elaborates on the novel design and application of the stochastic numerical computing method to analyze the fluid dynamics boundary layer problem represented by stiff non-linear, Falkner-Skan fluid (FSF) model. The quartic splines method (QSM) is designed to discretize the differential system of FSF system while the hybrid heuristics is exploiting via strength of Genetic Algorithms (GAs), aided with the Active-Set (AS) technique, i.e., QSM-GAs-AS. The designed scheme QSM-GAs-AS is implemented for the solution of the FSF model for sundry scenarios based on the variation of a single parameter, out of the three involved parameters, namely wall mass transfer parameter, wall movement parameter, and stream-wise pressure gradient parameter. The FSF model is solved for five, ten, and fifteen splines successfully and the solution outcomes of the FSF model, are compared with a deterministic numerical solver, i.e., Adams numerical technique. The closer agreement of both the solution outcomes validated the worth and efficacy of the proposed QSM-GAs-AS technique. The solution outcomes revealed that the velocity profile is enhanced for higher pressure gradient parameter, transfer of mass parameter and wall movement parameter.

Keywords: Quartic Splines Method; Active-Set algorithms; Genetic-Algorithms; Falkner-Skan fluid model; Hybrid computing

1. Introduction

The applications of fluid dynamics are diversified in sundry areas like petroleum industries, crowd dynamics, weather prediction, traffic engineering, and aerospace systems [1-3]. Fluid dynamics is basically based on the studies of pressure, energy, velocity, concentration, and temperature. The mathematicians have to cope with the applications of physical aspects so that to clear the blurring aspects of scientific disciplines related to fluid dynamics. One of the basic systems in fluid dynamics applications [4-7] is the Falkner-Skan fluid (FSF) model, which was instigated by Falkner and Skan in 1931 while studying viscous fluid submerged to flow in excess of a static wedge [8]. The similarity transformations are utilized, normally, to transform the PDEs into the equivalent non-linear ODEs of the third order, and further, it is analyzed for understanding its various dynamics [9-12].

Owing to the significance of the FSF model, various scientists have developed sundry analytical and numerical solvers to solve the system. A brief survey about the FSF model containing necessary comments is presented in Table 1. Although various deterministic techniques have been developed for numerical analysis of the FSF model, the stochastic solvers are attractive alternative options due to their ease of operation, simplicity of concept, reliability, and robustness for solving the FSF model non-linear differential systems of both ordinary and fractional order. Table 2, is provided for the reason to elaborate the applications based on stochastic solvers to the models based on differential systems. The stochastic computing solvers based on soft-computing techniques, as presented in Table 2, have proved their worth regarding convergence, efficacy, and consistent convergence. The soft-computing-based solvers have robust and efficient optimization in many fields of science, such as enhancing power management [13], binary methods optimization [14], the mathematical model of logistic infrastructure optimization [15], and fuzzy-control servo systems tuning optimization [16].

The literature survey of the FSF model as provided in Table 1, and stochastic computing techniques optimized with biological inspired soft-computations as given in Table 2, it can be deciphered that the artificial intelligence (AI) based computing solver of quartic splines method has not been applied for analyzing the dynamics of fluid flow model of Falkner-Skan with wall stretching and mass transfer effects. Therefore, it is aimed in this study to exploit the strength of stochastic computing paradigm of AI algorithms, i.e., the quartic splines method for formulations of merit function and optimized with Genetic Algorithms hybrid with Active-Set algorithms (QSM-GAs-AS) for the solution of FSF model.

Table 1. A brief history of Falkner-Skan fluid model.

Period	Description

1931-1971	Instigation of the Falkner Skan model was introduced in 1931 by numerically treating the boundary layer system [8]. In 1937, the most first physical problem of FSF was solved [17], while irregularities were identified in 1953 [18]. A reliable solution of the FSF was reported by Hertree's in 1966 [19]. The formulation of the first boundary value problems Falkner-Skan model was presented in 1970 [20], The non-linear FSF existence theorem was presented in 1971 [21]. In 1971 shooting numerical technique was exploited for approximate solution of the FSF model [22]. During this era, comparatively less literature for FSF is available regarding the numerical investigation of the system.
1972-2000	In this time of period, a lot of numerical methods were exploited for the numerical treatment of FSF. A few methods are briefly discussed here, which include Hermitian type finite-difference method introduced in 1978 [23], a quintic spline collocation method was reported in 1981 [24], a parametric differentiation based pseudo-spectral technique was introduced in 1984 [25], simulation study based on random vortex concept was introduced in 1989 [26], a finite difference with co-ordinates transformation method based on reduced input method was reported in 1998 [27], while in 1999, transformed Navier-Stokes technique was established for FSF [28].
2001-to date	The FSF is used as a standard model for analyzing the efficacy of analytical and numerical techniques. On the one hand, plenty of novel deterministic schemes have been introduced like Hankel-Padé method [29], the homotopy analysis method [30], Chebyshev collocation technique [31], optimal homotopy asymptotic method [32], Hermite functions pseudo-spectral method [33], Sinc-collocation method [34], eigenfunction based iterative methods [35], Fourier series-based methods [36]. While on the other hand, the Falkner-Skan flow involving in sundry fluid dynamics models including nanofluid flow investigation with magnetic field [37], magnetohydrodynamics (MHD) based flow with deceleration [38], fluid passing through a stretched surface with blowing/suction effects [39], analysis of the analysis of Casson model over a wedge [40], and many others fields. In all the above-mentioned investigations, the application of deterministic numerical techniques was reported to analyze the differential models.

Table 2. A brief history of stochastic solvers based on Neural-Networks.

Period	Description
1990-2000	The stochastic solvers based on neural networks (NNs) were applied in 1990 for the first time for the solution of finite difference equations [41]; the solution of differential equations by application of neural algorithms was a pioneer work of Fernandez and Meade in 1994 [42, 43]. Later on,

	partial differential equations were solved through neural networks models in 1998 [44], while in 2000, the multilayer networks were formulated [45].
2001-2010	During the said period, various stochastic numerical techniques were instigated for solving effectively the non-linear systems of differential equations such as radial based NNs [46], unsupervised NNs [47], unsupervised neural-fuzzy model [48], multilayer perceptron neural networks [49], cellular NNs [50], NNs satisfying exactly the arbitrary boundary conditions [51] and fractional neural networks [52].
2010-todate	In this era, the emphasis on stochastic solvers hybrid with global and local search algorithms were extensively exploited to numerically investigate governing systems arising in various fields of science such including fuzzy differential equations [53], the solids with electrical conduction [54], Navier-Stocks system [55], combustion theory-based fuel ignition model [56], non-linear pantograph differential system [57], multi-walled carbon nanotubes based nanofluid model [58], system of Volterra integral equation [59], Fredholm integral equation [60], fractional system of optimal control [61], Flierl–Petviashivili non-linear singular system [62], magnetohydrodynamics [63], Bratu problems [64], non-linear nanofluid flow of Jeffery-Hamel model [65], problem related to electromagnetic theory [66], fractional-order non-linear Riccati differential equation of [67], plasma physics related Troesch’s problem [68, 69], third grade fluid flow of thin film [70] and Bagley-Torvik model having fractional order [71].

The non-linear third-order FSF model with the transfer of mass along with the wall stretching effects is presented in the form of ODEs as:

$$\begin{aligned} & f''' + f f'' + \beta(1 - f'^2) = 0, \\ & f(0) = \gamma, \ f'(0) = \lambda, \ f'(1) = 1. \end{aligned} \tag{1}$$

In which the solution is presented $f(\eta)$, the first is given by $f'(\eta)$, $f''(\eta)$ stands for the second derivative, and the expression $f'''(\eta)$ represents the third derivative term with respect to the dimensionless independent variable η. The involved physical dimensionless parameters λ, β and γ represent wall movement, mass transfer, and stream-wise pressure gradient parameters, respectively.

Remarkable points of the present study are given as:

- ❖ The stochastic analysis is made by a novel application of Quartic Splines Method based on Genetic Algorithm hybrid with Active-Set scheme to analyze the FSF model.
- ❖ Highly reliable, qualitative outcomes and superior precision are benchmarks of the proposed QSM-GAs-AS stochastic scheme.

- Solution for five, ten, and fifteen splines were derived. The results were compared with the deterministic, Adams numerical technique, which validates the reliability, precision, and accuracy of the proposed methodology.
- The physical behavior of the involved parameters was observed, which describes that for higher values of all the parameters, wall movement, the stream wise pressure gradient and mass transfer have enhancing/increasing effects on the fluid velocity profile.

The rest of the manuscript is arranged as the procedure of the designed QSM-GAs-AS scheme given for the solution of FSF model and learning of weights for local search algorithms, the Active-Set (AS) and global search Genetic Algorithms (GAs) are presented in section 2, numerical experimentation and fitness function formulation provided in section 3, the simulation outcomes through graphical representations are provided in section 4, the concluding results are given in section 5, and the future recommended works are provided in the last section of the manuscript.

2. Proposed methodology

This section describes the designed methodology, the Quartic Splines Method based on GAs hybrid with AS (QSM-GAs-AS) algorithm, to solve the Falkner-Skan fluid flow problem by utilization of the stochastic methodology. The designed methodology has two main phases; the first phase presents the fitness function for the FSF model, while in the 2nd phase, the necessary procedure of the Genetic algorithms hybrid with Active Set algorithms is presented. The corresponding flowchart diagram explanations are provided in Fig. 1 and Fig. 2, respectively.

2.1. Mathematical formulation of Falkner-Skan equation

Mathematical modeling to solve the non-linear 3rd order Falkner-Skan fluid flow problems is provided in this section. The Quartic Splines method aims to achieve piecewise interpolating polynomials functions involving the 1st, 2nd, and 3rd order differential terms, which will accordingly provide solution representation function. Quartic Splines method is used for the Falkner-Skan fluid model based on the non-linear differential equation to obtain outcomes with necessary details as follows;

Firstly, the domain is divided into subintervals $[x_i, x_{i+1}]$ such that, $i = 1,2,3,\dots,n$ and to formulate splines information as under:

$$f(x) = \begin{cases} f1(x), & x_1 \le x \le x_2 \; ; \\ f2(x), & x_2 \le x \le x_3 \; ; \\ f3(x), & x_3 \le x \le x_4 \; ; \\ f4(x), & x_4 \le x \le x_5 \; ; \\ . & \\ . & \\ . & \\ f\ n(x), & x_n \le x \le x_{n+1} \, . \end{cases} \quad (2)$$

where $f(x)$ represented the desire solution for the FSF model while the i[th] spline $fi(x)$ represents with polynomial of degree four as:

$$f\ i\ (x) = a_i + b_i\ x + c_i\ x^2 + d_i\ x^3 + e_i\ x^4, \quad (3)$$

with 1st derivative is given as;

$$f'_i(x) = b_i + 2c_i x + 3d_i x^2 + 4e_i x^3, \tag{4}$$

second derivative as;

$$f''_i(x) = 2c_i + 6d_i x + 12e_i x^2, \tag{5}$$

and the third derivative as;

$$f'''_i(x) = 6d_i + 24e_i x. \tag{6}$$

The main objective of the study is to find the optimal solution of the Falkner-Skan equation by utilizing the GAs and AS algorithms based soft computing procedure. The unsupervised framework QSM is utilized through the universal estimated representation of FSF equation and optimization proficiency of GAs for finding the coefficient of each spline. The GAs scheme is broadly used for global search on number of optimization tasks [72-76]. The strength of GAs based optimization is enhanced by the procedure of integration with Active-Set (AS) algorithms [77, 78]. Hence, in the presented study, the leaning of the merit/objective function modeled with the QSM is conducted with GAs aided with AS, i.e., QSM-GAs-AS to solve the FSF equation.

Considering the general form of Falkner-Skan Eq. in the following way:

$$\Im(x, y, y', y'', y''') = 0, \tag{7}$$

the differential system in the spline model in the input interval can be represented for $i = 1, 2, 3, ..., p$;

$$\Im(x, f_i(x), f'_i(x), f''_i(x), f'''_i(x)) = 0, \quad x \in [x_i, x_{i+1}], \tag{8}$$

the last and first spline function must satisfy boundary conditions at ends,

$$f_1(x_1) = y_1, \tag{9}$$

whereas;

$$f_{n-1}(x_n) = y_n, \tag{10}$$

the function $f(x)$ necessarily is continuous, so:

$$f_i(x) = f_{i-1}(x) \text{ for } i = 2, 3, \ldots, p, \tag{11}$$

the first derivative of the function $f(x)$ necessarily be continuous, so:

$$f'_i(x) = f'_{i-1}(x) \text{ for } i = 2, 3, \ldots, p, \tag{12}$$

the second derivative of the function $f(x)$ necessarily be continuous, so:

$$f''_i(x) = f''_{i-1}(x) \text{ for } i = 2, 3, \ldots, p, \tag{13}$$

and the third derivative of the function $f(x)$ necessarily be continuous, so:

$$f'''_i(x) = f'''_{i-1}(x) \text{ for } i = 2, 3, \ldots, p, \tag{14}$$

2.2. Optimization paradigm with soft computing

The soft computing knacks and infrastructure is of immense importance for learning procedures of stiff optimization task and generally implemented with AI based algorithms. The utilization of soft computing can solve stiff differential systems based on linear and non-linear models. General procedure for applying the soft computing algorithms is given below:

(a). Startup or Initialization

The candidate or desire solution is represented with individual that are produced/generated arbitrarily with random number sequence, to defined the decision variable of the model.

(b). Fitness evaluations

Fitness evaluations is performed on a merit function in such a manner that solution of the problem approaches to the optimal solution. The fitness of all individual or candidate solutions is ranked with the help of fitness/objective function.

(c). Selection

The coherent, efficient and effective procedure is devised for appropriate selection of the candidate solution of optimization task such that a close agreement is achieved with respect to a optimal solution as compared to the worst solutions.

(d). Recommendation

Solutions having greater fitness are produced as pairwise adjusted potential solutions. As per the procedure developed by Goldberg, the offspring under the process should not be similar to the parents, but traits will be transferred in a unique way.

(e). Mutations

The replacement of new genes, which were unremembered in the recombination procedure and production of new offspring, then randomly modified by the operator of the recombination on the parental chromosome. Out of several variants, the individual traits with new alterations for the optimal solution is the most influential mutation technique.

(f). Replacements

In Genetic Algorithms, the variants of replacement operators depend on elitist replacement, generation-wise replacement, and time-dependent replacements. The parental papulation is replaced by creating the offspring through recombination, selection, and mutation.

(g). Termination/stoppage criteria

The Genetic Algorithms process of the execution is terminated if the stopping conditions on fitness, tolerances, install generation limits, execution time limit or total generations are satisfied.

(h). Local search-based fine-tuning with AS algorithm

When GAs selects appropriate individuals, they are fed to the Active-Set algorithms as a beginning stage for improvement and adjusting.

3. Numerical experimentation

The FSF model is treated by utilizing the QSM-GAs-AS scheme is for three scenarios, based on the three variants, i.e., mass transfer, wall movement, and stream-wise pressure gradient parameters for five, ten, and fifteen splines.

3.1. Five splines based QSM-GAs-AS scheme for Falkner-Skan model

The Falkner-Skan fluid flow model is represented as:

$$f''' + ff'' + \beta(1 - f'^2) = 0,\ f(0) = \gamma,\ \ f'(0) = \lambda,\ \ f'(1) = 1,\ x \in [0,1]. \qquad (15)$$

The objective function or figure of merit for the said five splines is given as follows:

$$\varepsilon = \sum_{k=0}^{7} \varepsilon_k, \qquad (16)$$

where

$$\varepsilon_0 = \sum_{i=1}^{5} \left(fi''' + fi\, fi'' + \beta\ (1 - fi'^2) \right), \qquad (17)$$

$$\varepsilon_1 = \sum_{i=1}^{5} \left(f_i - f_{i-1} \right)^2, \qquad (18)$$

$$\varepsilon_2 = \sum_{i=1}^{5} \left(f'_i - f'_{i-1} \right)^2, \qquad (19)$$

$$\varepsilon_3 = \sum_{i=1}^{5} \left(f''_i - f''_{i-1} \right)^2, \qquad (20)$$

$$\varepsilon_4 = \sum_{i=1}^{5} \left(f'''_i - f'''_{i-1} \right)^2, \qquad (21)$$

$$\varepsilon_5 = \sum_{i=1}^{5} \left(f_i(0) - \gamma \right)^2, \qquad (22)$$

$$\varepsilon_6 = \sum_{i=1}^{5} \left(f_i'(0) - \lambda \right)^2, \qquad (23)$$

$$\varepsilon_7 = \sum_{i=1}^{5} \left(f_i'(1) - 1 \right)^2, \qquad (24)$$

Similarly, the cost/objective function for ten and fifteen splines is constituted. The residual error (RE) for five splines is represented, mathematically as:

$$RE = \sum_{i=1}^{5} \left(f_{i,j}\,''' + f_{i,j}\, f_{i,j}\,'' + \beta\,(1 - f_{i,j}\,'^{2}) \right), \qquad (25)$$

where $f_{i,j}$ represents the approximate outcome of i^{th} sub-spline for the j^{th} input value with respect to step-size $h = 0.1$, for $x \in [0,1]$.

3.2. Ten splines based QSM-GAs-AS scheme for Falkner-Skan model

The fitness function for ten splines is constructed the same as for five splines by taking $i = 1$ to $i = 10$ in Eq. (25) for finding RE.

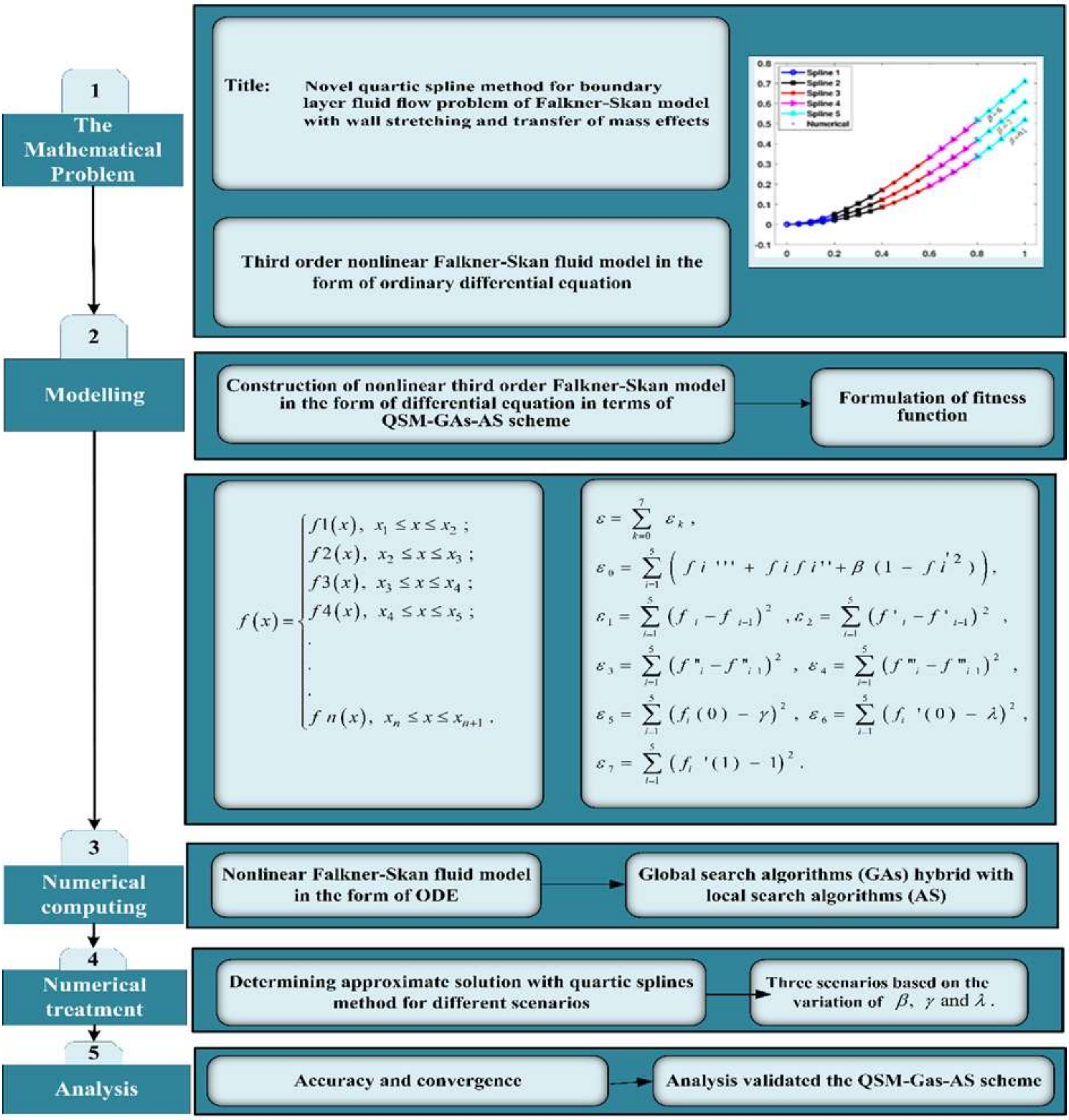


Figure 1. Flowchart diagram of Falkner-Skan fluid model and proposed methodology.

3.3. Fifteen splines based QSM-GAs-AS scheme for Falkner-Skan model

The fitness function for fifteen splines is constructed the same as for five splines by taking $i=1$ to $i=15$ in Eq. (25) for finding RE.

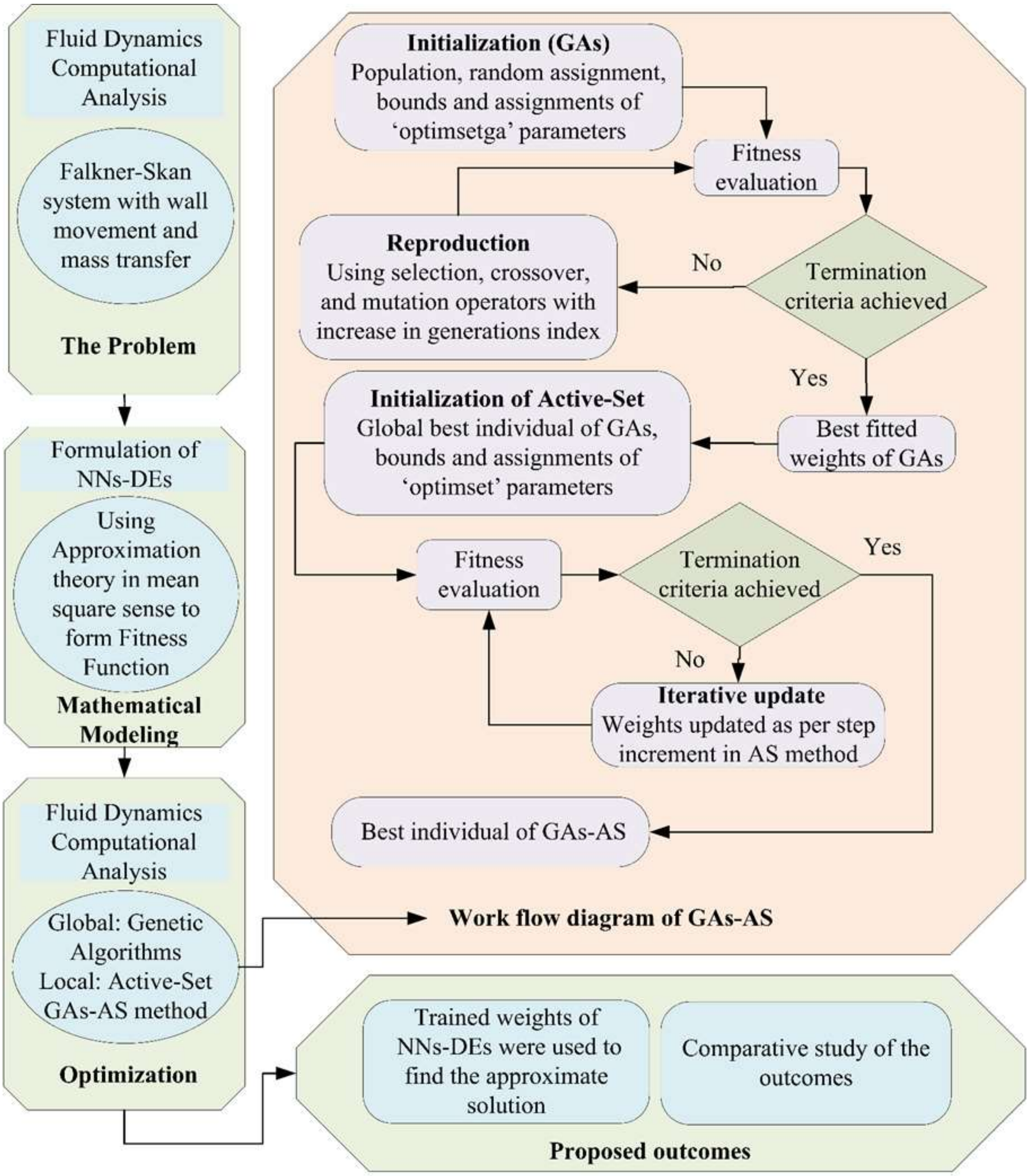


Figure 2. Flowchart representation of the QSM-GAs-AS methodology to solving Falkner-Skan non-linear third order fluid model.

Table 3: Setting of parameters for GA-AS scheme.

Search Method	**Description of parameters**	**Setting**	**Description of parameters**	**Setting**
Genetic Algorithm i.e. GA	Generations Tol Fun Stall Gen Limit Tol Con Fitness Limit	200 1e-20 75 1e-20	Creation Fcn Mutation Fcn Crossover Fcn Selection Fcn Population Size	Ga creation uniform mutation adapt feasible crossover heuristic selection stochunif 200
	PopIn it Range	[−1;1]	Elite Count	20
			Others	Defaults
Active-Set i.e. AS	Fin Diff Type Max	central	Algorithm	active-set
	Fun Evals	200000	Tol Fun	1e-35
	Tol Con	1e-35	Tol X	1e-30
	Max Iter	800	Start point Others	GAs best weights Defaults

Table 4: Description of involved parameter.

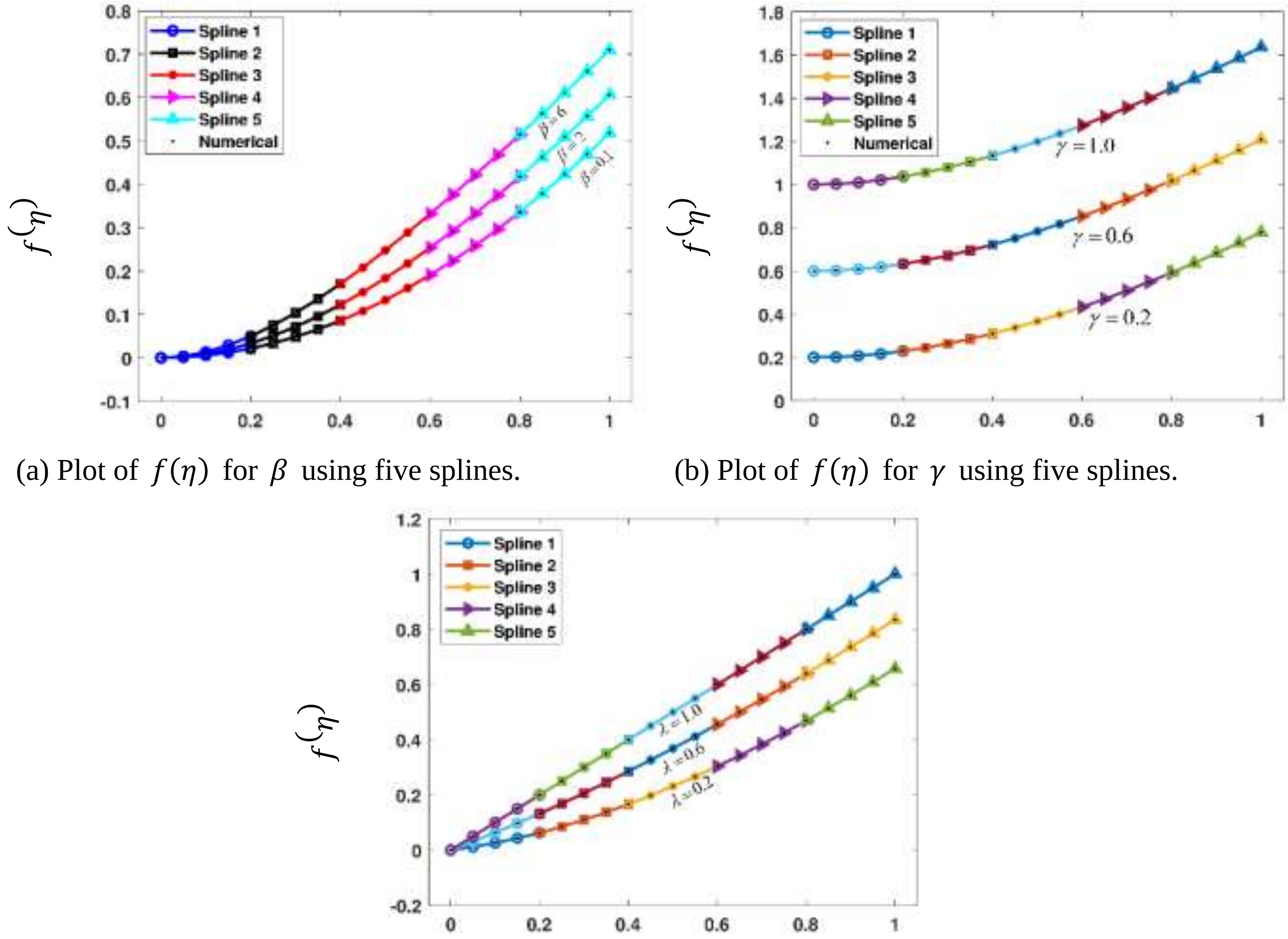
Parameter	$\boldsymbol{\beta}$	$\boldsymbol{\gamma}$	$\boldsymbol{\lambda}$
Case-1	0.1	0	0
Case-2	2	0	0
Case-3	6	0	0
Case-1	1	0.2	0
Case-2	1	0.6	0
Case-3	1	1	0
Case-1	1	0	0.2
Case-2	1	0	0.6
Case-3	1	0	1

4. Results, discussions and explanations

The results obtained, during the analysis of numerical experimentation for the Quartic-Splines-Method based on neural networks scheme optimized with Genetic Algorithm hybrid with Active-Set algorithm for five, ten and fifteen splines in Fig. 3, Fig. 4 and Fig. 5, respectively, during the solution of Falkner-Scan fluid model, are discussed in this section. The Falkner-Skan model is analyzed using the proposed technique QSM-GAs-AS for the variations of three involved parameters, which are the parameter of stream-wise pressure gradient (β), the parameter of mass transfer of the wall (γ) and stretching parameter of the wall (λ). All the three problems/variations are discussed here, separately. The comparison of results of the proposed stochastic Quartic-Splines-Method is carried out with Adams numerical technique, which is considered as a standard solution during the entire analysis. The variations of the physical parameters are displayed in Table 4.

4.1. Dynamics of FSF model for the variation of pressure gradient parameter β:

The variation of stream-wise pressure gradient parameter β is considered here by taking $\beta = 0.1, 2, 6$, while considering the fixed values wall mass transfer and wall stretching parameters as $\gamma = 0$ and $\lambda = 0$, respectively. The solution of Falkner-Skan fluid (FSF) model given in Eq. (1) is derived by exploiting the strength of stochastic numerical solver, the Quartic-Splines-Method, based on a global search scheme Genetic-Algorithm and local search Active-Set algorithms for five, ten, and fifteen splines. The solution of the proposed technique QSM-GAs-AS is compared with the standard numerical solution obtained through the Adams numerical technique. The outcomes of the proposed technique for the Falkner-Skan model along with comparison are displayed in subfigures Fig. 3(a) of Fig. 3, Fig. 4(a) of Fig. 4, and Fig. 5(a) of Fig. 5 for five, ten, and fifteen splines, respectively. These subfigures show the authenticity of the proposed technique QSM-GAs-AS, as this close agreement of the results of the proposed solution and that of the standard solution of Adams numerical method. Moreover, the outcomes revealed that a higher stream-wise pressure gradient boosts up the velocity of the fluid, as shown in subfigure (a) of Fig. 3, Fig. 4, and Fig. 5.



(a) Plot of $f(\eta)$ for β using five splines.

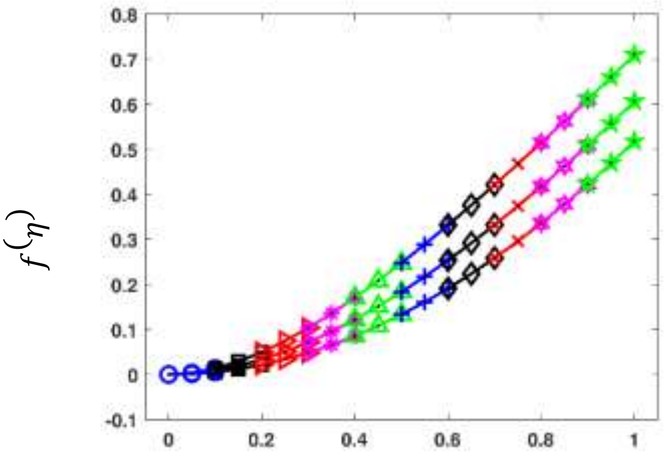
(b) Plot of $f(\eta)$ for γ using five splines.

(c) Plot of $f(\eta)$ for λ using five splines.

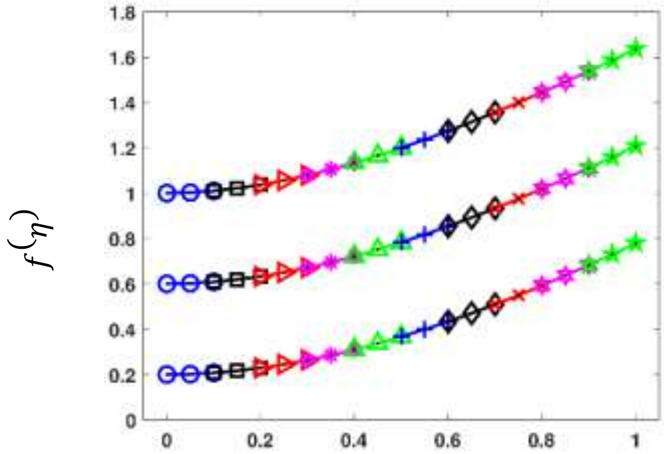
Figure 3. Solution plots of f(η) for five splines using QSM-GAs-AS.

4.2 Dynamics of FSF model for the variation of mass transfer of wall γ:

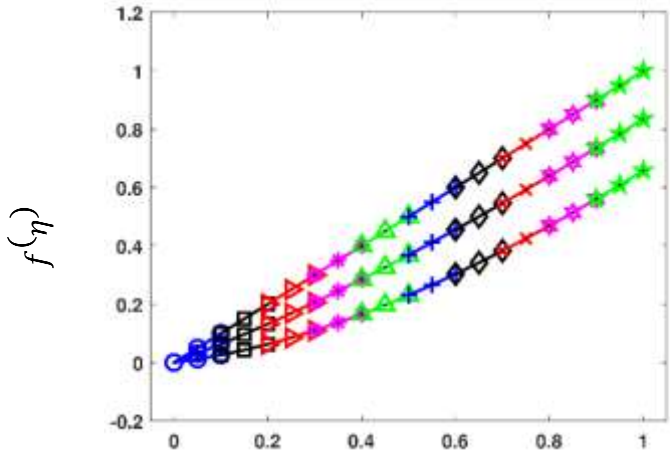
The variation of mass transfer of the wall γ is considered here by taking $\gamma = 0.2, 0.6, 1$, while considering the fixed values of stream-wise pressure gradient and wall stretching parameters as $\beta = 1$ and $\lambda = 0$, respectively. The solution of the FSF model given in Eq. (1), is derived by exploiting the strength of stochastic numerical solver, the QSM-GAs-AS scheme for five, ten, and fifteen splines, as shown in Figures 3-5. The solution of the proposed technique QSM-GAs-AS is compared with the standard numerical solution obtained through the Adams numerical technique. The outcomes of the proposed technique for the Falkner-Skan model, along with comparison, are displayed in subfigures Fig. 3(b) of Fig. 3, Fig. 4(b) of Fig. 4, and Fig. 5(b) of Fig. 5 for five, ten and fifteen splines, respectively. These subfigures show the worth of the proposed technique QSM-GAs-AS, as there is a close agreement between the proposed solution's results and that of the standard solution of the Adams numerical method. Furthermore, the outcomes revealed that with higher wall mass transfer parameter, the velocity of fluid increases, as shown in the subfigure (b) of Fig. 3, Fig. 4, and Fig. 5.



(a) Plot of $f(\eta)$ for β using ten splines.



(b) Plot of $f(\eta)$ for γ using ten splines.

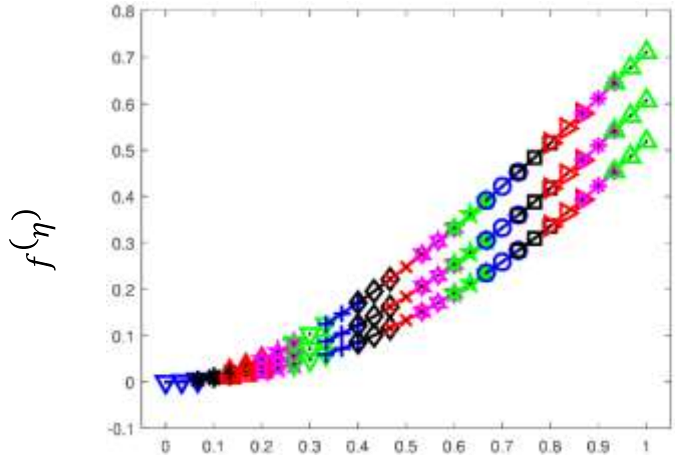


(c) Plot of $f(\eta)$ for λ using ten splines.

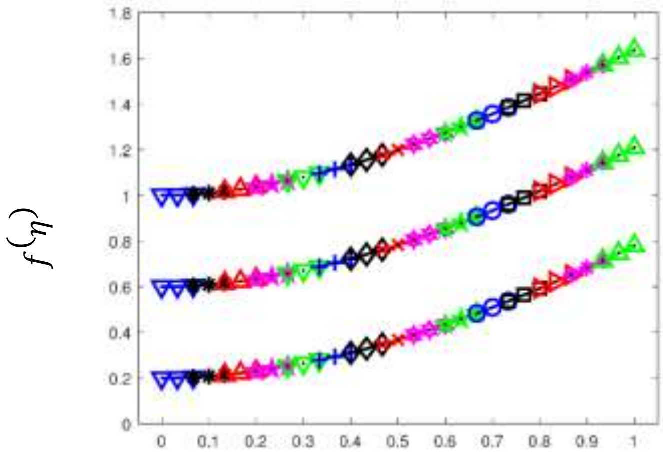
Figure 4. Solution plots of f(η) for ten splines using QSM-GAs-AS.

4.3 Dynamics of FSF model for the variation of stretching parameter of the wall λ:

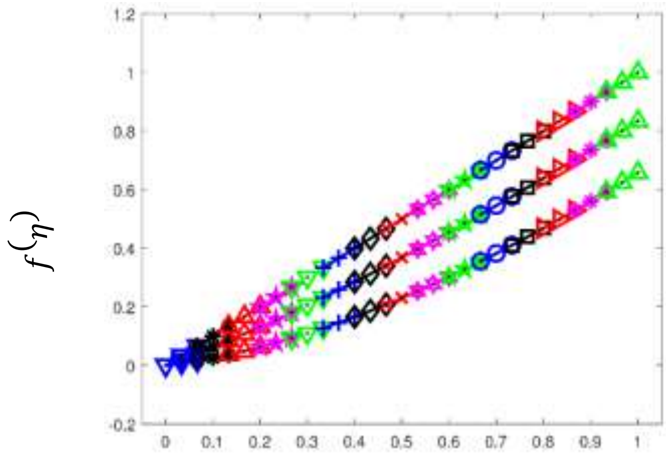
The variation of wall stretching parameter is considered here by taking $\lambda = 0.2, 0.6, 1$, while considering the fixed values of stream-wise pressure gradient and wall mass transfer parameters as $\beta = 1$ and $\gamma = 0$, respectively. The solution of the FSF model given in Eq. (1), is derived by utilizing the strength of stochastic numerical solver, the QSM-GAs-AS scheme for five, ten, and fifteen splines, as shown in Figures 3-5. The solution of the proposed technique QSM-GAs-AS is compared with the standard numerical solution obtained through the Adams numerical technique. The outcomes of the proposed technique for the Falkner-Skan model, along with comparison, are displayed in subfigures Fig. 3(c) of Fig. 3, Fig. 4(c) of Fig. 4 and Fig. 5(c) of Fig. 5 for five, ten, and fifteen splines, respectively. These subfigures show the worth of the proposed technique QSM-GAs-AS, as there is a close agreement between the proposed solution's results and that of the standard solution of Adams numerical method. Furthermore, the outcomes revealed that with higher wall stretching parameter, the velocity of fluid increases, as shown in the subfigure (c) of Fig. 3, Fig. 4, and Fig. 5.



(a) Plot of $f(\eta)$ for β using fifteen splines.



(b) Plot of $f(\eta)$ for γ using fifteen splines.



(c) Plot of $f(\eta)$ for λ using fifteen splines.

Figure 5. Solution plots of f(η) for fifteen splines using QSM-GAs-AS.

5. Conclusion

The main finding/inferences of the study are presented in brief as follows:

- The novel design and application of Quartic-Splines-Method, optimized with GAs, and AS algorithms is carried out to determine the solution of nonlinear Falkner-Skan fluidic model by varying the stream-wise pressure gradient, wall mass transfer, and wall stretching parameters for five, ten, and fifteen splines.
- The proposed stochastic numerical solver QSM-GAs-AS is applied by exploiting the strength of the global search scheme of the Genetic Algorithm and local search strength of Active-Set algorithms.
- The Falkner-Skan fluidic model is solved successfully by the proposed technique QSM-GAs-AS, and the outcomes are presented with enough graphical and numerical illustrations to prove its worth.
- The close agreement of the results of the standard solution and the solution of the proposed technique validated that the designed QSM-GAs-AS algorithm as a convergent, accurate, alternate and authentic solver developed through knacks of AI procedures.

6. Future work

In the future, the Quartic-Splines-Method based on Genetic-Algorithm hybrid with Active-Set scheme can be applied to solve other non-linear differential equations based complex fluidic models. The fluid flow problems with various effects such as magnetic field, radiation, heat convection, etc., can also be solved by the QSM-GAs-AS scheme. Moreover, it seems promising to investigate in applying the proposed scheme for solving diffusion models [79], closed geodesic problems [80], fractional order problems [81-85] and fluid flow [86] and nonlinear system identification [87-88].

Author Contributions: Conceptualization, M.A.Z.R.; methodology, M.S.; software, M.S., K.S.N; formal analysis, F.S., M.A., M.S.A, CAS and I.U.; resources, S.I., MS, MAS, CAS; writing—original draft preparation, F.S. and I.U.; writing—review and editing, N.I.C., M.A.Z.R., K.S. N, M.A and M.S.A; visualization, N.I.C. M.A.Z.R., CAS; supervision, S.I. and K.S.N; project administration, K.M.C. and A.H.M.; funding acquisition, M.A., M.S.A, CAS. All authors have read and agreed to the published version of the manuscript.

Conflicts of Interest: The authors declare no conflict of interest.

Acknowledgements: “The authors extend their appreciation to the Deanship of Scientific Research at King Khalid University (KKU) for funding this research project Number (R.G.P.2/133/43). This study is supported via funding from Prince Sattam bin Abdulaziz University project number (PSAU/2023/R/1444) "

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